2/2/23 MATH 2060 A Tutorial

Announcements:

- HW2 Due Feb 10 11:00am.

Section 6.3. L'Hopital's Rule:

Want to be able to handle

$$\lim_{x \to a} \frac{f(x)}{g(x)} \qquad \frac{0}{0}, \ \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad 0^0, \ldots$$

Thm 6.3.1 f, g defined on $[a,b]$, $f(a) = 0 = g(a)$, $g(x) \neq 0$ on (a,b). If f, g differentiable at a, with $g'(a) \neq 0$, then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

L'Hopital's Rule I: (Thm 6.3.3) f, g differentiable on (a,b), $g'(x) \neq 0$ for all $x \in (a,b)$ w/ $\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x)$, then

If $\lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)}$ exists or is $+\infty, -\infty$, then $\lim\limits_{x\to a^+} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)}$.

L'Hopital's Rule II: f, g differentiable on (a,b), $g'(x) \neq 0$ for all $x \in (a,b)$

$\lim\limits_{x\to a^+} g(x) = +\infty$, then

If $\lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)}$ exists or is $+\infty, -\infty$, then $\lim\limits_{x\to a^+} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)}$.

Also work for left-handed limits, two-sided limits or if $a = -\infty$, $b = +\infty$.

Quick Examples

- $\lim\limits_{x\to\infty} \dfrac{\ln x}{x} \overset{\text{LHR}}{=} \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{1} = 0$.

  "$\dfrac{\infty}{\infty}$"

- $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right) = \lim\limits_{x\to 0^+} \dfrac{\sin x - x}{x \sin x} \overset{\text{LHR}}{=} \lim\limits_{x\to 0^+} \dfrac{\cos x - 1}{\sin x + x\cos x} \overset{\text{LHR}}{=} \lim\limits_{x\to 0^+} \dfrac{-\sin x}{2\cos x + x\sin x} = \dfrac{-0}{2} = 0$.

  "$\infty - \infty$" "$\dfrac{0}{0}$"

$\cdot \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} \overset{LHR}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0.$

"$0\cdot(-\infty)$" "$\frac{-\infty}{\infty}$"

Q3: $f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right), & x\in(0,1] \\ 0, & x=0 \end{cases}$, $g(x) = x^2$ for $x\in[0,1]$

Show that $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x)$ but that $\lim_{x\to 0} \frac{f(x)}{g(x)}$ DNE. ← can't apply Thm 6.3.1, Thm 6.3.3.

Pf: Clearly $\lim_{x\to 0} g(x) = \lim_{x\to 0} x^2 = 0$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2\sin\left(\frac{1}{x}\right) = 0$ Squeeze Thm $\left(|f(x)| \le x^2\right)$.

$\frac{f(x)}{g(x)} = \frac{x^2\sin\left(\frac{1}{x}\right)}{x^2} = \sin\left(\frac{1}{x}\right)$ so $\lim_{x\to 0} \frac{f(x)}{g(x)}$ DNE: Choose $x_n := \frac{1}{2n\pi} \to 0$ as $n\to\infty$

$y_n := \frac{1}{\left(\frac{\pi}{2}+2n\pi\right)} \to 0$ as $n\to\infty$

Cannot apply Thm 6.3.1 since $g'(0) = 0$.

Cannot apply Thm 6.3.3 since $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ DNE:

$$\frac{f'(x)}{g'(x)} = \frac{2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)}{2x} = \sin\left(\frac{1}{x}\right) - \frac{\cos\left(\frac{1}{x}\right)}{2x}.$$

again, use $x_n := \frac{1}{2n\pi}$

$y_n := \frac{1}{\left(\frac{\pi}{2} + 2n\pi\right)}$

to show $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ DNE.

Q4: $f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$, $g(x) = \sin x$

Use Thm 6.3.1 to show $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$. Explain why Thm 6.3.3. cannot be used.

Pf: Clearly $f(0) = 0 = g(0)$. $g(x) \neq 0$ for $x > 0$ and x small enough.

$g'(0) = \cos 0 = 1 \neq 0$.

So by Thm 6.3.1, $\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)} = f'(0)$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$$

Let $\varepsilon > 0$, then since $\left|\frac{f(x)}{x}\right| \leq \frac{x^2}{x} = x$, as long as $|x| < \varepsilon$, we have

$\left|\frac{f(x)}{x}\right| \leq |x| < \varepsilon$. So $f'(0) = \lim_{x \to 0} \frac{f(x)}{x} = 0$

Cannot apply Thm 6.3.3 since $f'(x)$ DNE for all $x \neq 0$.

Let $a \neq 0$, then $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \begin{cases} \lim_{x \to a} \frac{f(x) - a^2}{x - a}, & a \in \mathbb{Q} \\ \lim_{x \to a} \frac{f(x)}{x - a}, & a \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

In both cases, since $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$ dense in $\mathbb{R}$, can pick $\{x_n\} \subseteq \mathbb{Q}$

$x_n \to a$ as $n \to \infty$, $\{y_n\} \subseteq \mathbb{R} \setminus \mathbb{Q}$, $y_n \to a$ as $n \to \infty$.

Check $\lim_{n \to \infty} \frac{f(x_n) - f(a)}{x_n - a} \neq \lim_{n \to \infty} \frac{f(y_n) - f(a)}{y_n - a}$.

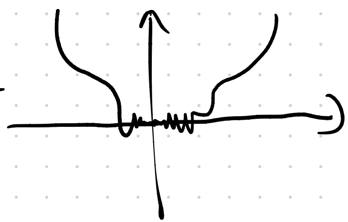
Q5: $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$, $g(x) = \sin x$.

Show that $\lim_{x \to 0} \frac{f(x)}{g(x)} = 0$, but that $\lim_{x \to 0} \frac{f'(x)}{g'(x)}$ DNE.

Pf: $\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{\sin x} = \lim_{x \to 0} \frac{x}{\sin x} x \sin\left(\frac{1}{x}\right) = \cancel{\underbrace{\lim_{x \to 0} \frac{x}{\sin x}}_{\to 1}} \cancel{\underbrace{\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)}_{\to 0 \text{ squeeze thm.}}} = 0.$

$$\frac{f'(x)}{g'(x)} = \frac{2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)}{\cos(x)} = \frac{2x \sin\left(\frac{1}{x}\right)}{\cos(x)} - \frac{\cos\left(\frac{1}{x}\right)}{\cos x}$$

$\frac{f(x)}{g(x)}$ graph



$x_n := \frac{1}{2n\pi} \to 0 \quad n \to \infty$

$y_n := \frac{1}{\left(\frac{\pi}{2} + 2n\pi\right)} \to 0 \text{ as } n \to \infty$